Subharmonic functions. Maximum Principle. The reflection principle.

Tuesday, March 23, 2021 11:00 AM

<u>Def.</u> Let $v \in C(\Omega)$ (real-valued). (but can be $v(z)=-\infty$) v is called <u>subharmonic in Ω</u> if $\forall z_0 \in \Omega$, $\exists r_{z_0} < dist(z_0, \partial\Omega)$ ($B(z_0, r_{z_0}) \subset \Omega$) we have:

$$v(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} v(z_0+re^{it})\,dt \quad \forall r \le r_{z_0}.$$

<u>Remark.</u> We will show it holds for <u>any</u> $r < dist(z_0, \partial\Omega)$.

<u>On definition of integral:</u> $\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{it})\,dt = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} u_n(z_0+re^{it})\,dt$,

where $u_n(z) := \max(-n, u(z))$. lim always exists - decreasing!

<u>Examples.</u> 1) If u is harmonic, then $u, -u$ - both subharmonic.

2) If $f \in A(\Omega)$, then

$$|f(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|\,dt.$$

So $|f|$ is subharmonic.

3) If $f \in A(\Omega)$ then $\log|f(z)|$ is subharmonic.

<u>Proof.</u> If $f(z_0) \ne 0$, then $\exists B(z_0, r_1)$: $|z-z_0| \le r_1 \Rightarrow f(z) \ne 0$.

So $\log f$ is well-defined in $B(z_0, r_1)$, $\log|f| = Re(\log f) \in Harm(B(z_0, r_1))$

So $\log|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(z_0+re^{it})|\,dt \quad \forall r < r_1$.

If $f(z_0) = 0$, then $\exists B(z_0, r): 0 < |z-z_0| < r \Rightarrow f(z) \ne 0$.

So $\log|f(z_0)| = -\infty < \frac{1}{2\pi}\int_0^{2\pi} \log|f(z_0+re^{it})|\,dt$

<u>Theorem</u> (Maximum principle).

Let v be subharmonic in Ω. Assume that v reaches maximum at $z_0 \in \Omega$. Then $v \equiv const$.

<u>Remark.</u> When $v = |f|$, $f \in A(\Omega)$ - the usual Maximum principle (slightly weaker).

<u>Proof.</u> Let $v(z_0) = M$. Let $\Omega_1 = \{z \in \Omega : v(z) = M\}$

$\Omega_2 = \{z \in \Omega : v(z) < M\}$.

Ω_2 is open - since v is continuous.

Let $z \in \Omega_1$. Then let us prove that $B(z, r) \subset \Omega_1$ for any $r < r_z$. This would imply Ω_1 - open.

Indeed, if $r' := |w_0 - z| < r$, $v(w_0) < M$, then $\exists \varepsilon > 0$: $|w - w_0| < \varepsilon \Rightarrow v(w) < M - \varepsilon$.

Then $M = v(z) \le \frac{1}{2\pi}\int_0^{2\pi} v(z+r'e^{it})\,dt \le \frac{1}{2\pi}\left(M(2\pi - \varepsilon) + \varepsilon(M-\varepsilon)\right) \le M - \frac{\varepsilon^2}{2\pi}$ - contradiction!

So $\Omega_1 \cup \Omega_2 = \Omega$, $\Omega_1 \neq \emptyset$, $\Omega_1 \cap \Omega_2 = \emptyset \Rightarrow \Omega = \Omega_1$ (connected).

So $v \equiv M$ ∎

Corollary. Let v be continuous on bounded and closed S, subharmonic on $\mathrm{Int}(S)$. Then $\max_{z \in S} v(z) = \max_{z \in \partial S} v(z)$.

Corollary (Harmonic majoration). If $u, v \in C(S)$ (S - closed, bounded), $u \in \mathrm{Harm}(\mathrm{Int}\, S)$, v - subharmonic on $\mathrm{Int}(S)$, $u(z) \geq v(z)$ $\forall z \in \partial S$.

Then $\forall z \in S$: $u(z) \geq v(z)$.

Proof. $v - u$ is subharmonic in $\mathrm{Int}(S)$,

$$\max_{z \in S}(v(z) - u(z)) = \max_{z \in \partial S}(v(z) - u(z)) \leq 0 \quad ∎$$

Corollary. Let u be subharmonic in Ω. Then $\forall z_0 \in \Omega$ $\forall r < \mathrm{dist}(z_0, \partial\Omega)$: $u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt$

Proof. Let, as before, $u_n(z) = \max(-n, u(z)) \in C(\Omega)$

Then let $V_n(z)$ be the Poisson integral of u_n in $B(z_0, r)$.

By Majoration, since $\forall z \in \partial B(z_0, r)$, $u_n(z) = V_n(z)$, we have

$$u(z_0) \leq u_n(z_0) \leq V_n(z_0) = \frac{1}{2\pi}\int_0^{2\pi} V_n(z_0 + re^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi} u_n(z_0 + re^{it})\,dt$$

Now take $\lim_{n \to \infty}$. Get

$$u(z_0) \leq \lim_{n \to \infty} \frac{1}{2\pi}\int_0^{2\pi} u_n(z_0 + re^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt \quad ∎$$

Theorem. Assume both u and $-u$ are subharmonic in Ω (i.e. $\forall z \in \Omega$, $\exists r_z : r < r_z$: $u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{it})\,dt$ - mean value property and $u \in C(\Omega)$).

Then $u \in \mathrm{Harm}(\Omega)$. In particular, $u \in C^\infty(\Omega)$.

Proof. We need to show: $\forall z \in \Omega$ $r < r_z$, $u \in \mathrm{Harm}(B(z, r))$, since then $u \in C^2(\Omega)$, $\Delta u = 0$.

Let U be the Poisson integral of $u(re^{it} + z)$ in $B(z, r)$:

$$U(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |w - z|^2}{|w - z - re^{it}|^2}\,dt.$$

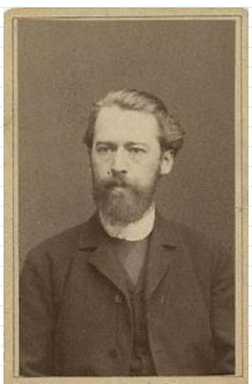
Then $U, u \in C(\overline{B(z,r)})$ ($u \in C(\Omega)$, U - by Schwarz).
On $\{|w-z|=r\}$: $U(w)=u(w)$ (by Schwarz!)
On $B(z,r)$: $U-u$ and $u-U$ are subharmonic.
So, by maximum principle, $\forall w \in B(z,r)$:

$$U(w)-u(w) \le \max_{w\in\{|w-z|=r\}} (U(w)-u(w)) = 0$$

$$u(w)-U(w) \le 0.$$

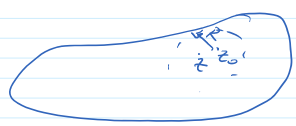
So in $B(z,r)$, $u=U$. But U is harmonic in $B(z,r)$ ∎

Theorem (Uniqueness of Dirichlet solution).
Let Ω be bounded, $u, v \in C(\overline{\Omega})$, $u \equiv v$ on $\partial\Omega$.
Then $u \equiv v$ on $\overline{\Omega}$. $u, v \in Harm(\Omega)$
Proof. Apply Maximum Principle to $u-v$ and $v-u$ ∎



Axel Harnack

Theorem (Harnack inequality)
Let $u \in Harm(\Omega)$, $u \ge 0$. Let $z, z_0 \in \Omega$, $|z-z_0| < dist(z_0, \partial\Omega) =: R$
Then $\frac{R-|z-z_0|}{R+|z-z_0|} u(z_0) \le u(z) \le \frac{R+|z-z_0|}{R-|z-z_0|} u(z_0)$



Proof. Let $|z-z_0| < R' < R$. By Schwarz Theorem:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R')^2-|z-z_0|^2}{|R'e^{it}-(z-z_0)|^2}\, u(z_0+R'e^{it})\,dt$$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+R'e^{it})\,dt.$$

Note that $(R'+|z-z_0|)^2 \ge |R'e^{it}-(z-z_0)|^2 \ge (R'-|z-z_0|)^2$

So $\frac{R'-|z-z_0|}{\ldots}$ $\quad (R')^2-|z-z_0|^2 \quad R'+|z \ldots$

Note that $(R'+|z-z_0|)^2 \geq |R'e^{it}-(z-z_0)|^2 \geq (R'-|z-z_0|)^2$

So $$\frac{R'-|z-z_0|}{R'+|z-z_0|} \leq \frac{(R')^2-|z-z_0|^2}{|Re^{it}-(z-z_0)|^2} \leq \frac{R'+|z-z_0|}{R'-|z-z_0|}$$

So $$u(z) \geq \frac{1}{2\pi}\int_0^{2\pi} \frac{R'-|z-z_0|}{R'+|z-z_0|} u(z_0+R'e^{it})\,dt = \frac{R'-|z-z_0|}{R'+|z-z_0|} u(z_0).$$

Same way $u(z) \leq \frac{R'+|z-z_0|}{R'-|z-z_0|} u(z_0)$.

Now let $R' \to R$ to obtain Harnack inequality ∎

Theorem (Harnack Principle).

Let Ω be a region, Ω_n - sequence of regions such that $\forall z \in \Omega$ $\exists r_z, N_z$: $n \geq N_z \Rightarrow B(z, r_z) \subset \Omega_n$. Let $u_n \in Harm(\Omega_n)$,

Assume if $|\zeta - z| < r_z$, $n > m \geq N_z \Rightarrow u_n(\zeta) \geq u_m(\zeta)$.

Then either:

1) $u_n(z) \to \infty$ locally uniformly on Ω.

2) $u_n(z) \to u(z)$ locally uniformly, $u \in Harm(\Omega)$.

Proof. Since $\forall z \in \Omega$, $(u_n(z))$ is increasing for $n \geq N_z$.

$\exists \lim_{n\to\infty} u_n(z) =: u(z)$

Observe: $u_n(\zeta) - u_1(\zeta) \geq 0$ for $n \geq N_z$, $\zeta \in B(z, r_z)$.

So $\frac{r_z+|\zeta-z|}{r_z-|\zeta-z|}(u_n(z)-u_{N_z}(z)) \geq \underbrace{u_n(\zeta)-u_{N_z}(\zeta)}_{} \geq \frac{r_z-|\zeta-z|}{r_z+|\zeta-z|}(u_n(z)-u_{N_z}(z))$

Thus if $u(z)=\infty$ then $u(\zeta)=\infty$ $\forall \zeta \in B(z, r_z)$.

$\left(u_n(z) \to \infty \Rightarrow u_n(\zeta) \geq u_1(\zeta) + \frac{r_z-|\zeta-z|}{r_z+|\zeta-z|}(u_n(z)-u_1(z)) \to \infty\right)$.

if $u(z)<\infty$ then $\forall \zeta \in B(z, r_z)$ $u(\zeta)<\infty$.

So $\left.\begin{array}{l}\Omega^1 := \{z \in \Omega : u(z) < \infty\} \\ \Omega^2 := \{z \in \Omega : u(z) = \infty\}\end{array}\right\}$ open.

So either:

Case 1 $\Omega^1 = \emptyset$. Then $u_n(z) \to u(z)$, locally uniformly on $B(z, r_z)$, so locally uniformly on Ω.

Case 2. $\Omega^2 = \emptyset$. If $n > m > N_z$, $\zeta \in B(z, r_z)$, we have

$$\underbrace{u_n(\zeta)-u_m(\zeta)}_{\substack{\text{uniformly} \\ \text{Cauchy in } B(z,r') \\ \forall r' < r_z.}} \leq \frac{r_z+|\zeta-z|}{r_z-|\zeta-z|}\underbrace{(u_n(z)-u_m(z))}_{\text{Cauchy}}$$

So $u_n \to u$ locally uniformly so $u \in C(\Omega)$ and

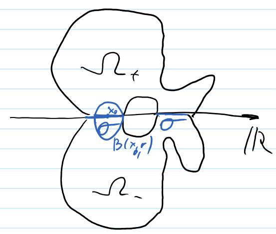
$$u(z)=\lim_{n\to\infty} u_n(z) = \lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} u_n(z+re^{it})\,dt \underset{\text{uniform on } \{|\xi-z|=r\}}{=} \frac{1}{2\pi}\int_0^{2\pi}\lim_{n\to\infty} u_n(z+re^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{it})\,dt.$$

$n\ge N_z$, $r<r_z$.

So $u\in \mathrm{Harm}(\Omega)$ ■

An application: the Reflection Principle.

<u>Theorem</u> (Schwarz).



Let $\Omega^+\subset \mathbb{H}=\{z: \mathrm{Im}\, z>0\}$, $\sigma:=\partial\Omega^+\cap\mathbb{R}$, $\Omega=\Omega^+\cup\sigma\cup\Omega^-$,
where $\Omega^-=\{z:\bar z\in\Omega^+\}$

Let $v\in \mathrm{Harm}(\Omega^+)$, $v\in C(\Omega^+\cup\sigma)$, $v(x)=0$ $\forall x\in\sigma$.

Then $\tilde v=\begin{cases} v(z), & z\in\Omega^+\\ 0, & z\in\sigma\\ -v(\bar z), & z\in\Omega^-\end{cases} \in \mathrm{Harm}(\Omega)$.

If $f\in A(\Omega^+)\cap C(\Omega^+\cup\sigma)$ $v=\mathrm{Im}\, f$ — as above ($f(x)\in\mathbb{R}$ $\forall x\in\sigma$), then
$$\tilde f(z)=\begin{cases} f(z), & z\in\Omega^+\cup\sigma\\ \overline{f(\bar z)}, & z\in\Omega^-\end{cases} \in A(\Omega).$$

<u>Proof</u>. We just need to check that $\tilde v$ is harmonic on σ.

Let $x_0\in\sigma$. Take $r<\mathrm{dist}(x_0,\partial\Omega)$. Let $P_{\tilde v}$ be the Poisson integral of $\tilde v(x_0+re^{it})$ in $B(x_0,r)$. By symmetry, $P_{\tilde v}(x)=0$ $\forall x\in\sigma$.

So, on $\partial(B(x_0,r)\cap\mathbb{H})$: $\begin{cases} P_{\tilde v}(x)=v(x)=0 & (x\in\sigma)\\ P_{\tilde v}(x)=v(x) & (x\notin\sigma)\end{cases} \underset{\text{Maximum principle}}{\Longrightarrow} \forall z\in\overline{B(x_0,r)\cap\mathbb{H}},\quad v(z)=P_{\tilde v}(z).$

Again, by symmetry, $\forall z\in\overline{B(x_0,r)\cap\mathbb{H}_-}$ ($\mathbb{H}_-=\{z:\mathrm{Im}\,z<0\}$),
$P_{\tilde v}(z)=\tilde v(z)$. So $\tilde v$ is harmonic at x_0.

For the second part, $f=u+iv$, let $\tilde f=\tilde u+i\tilde v$.

By first part, $\tilde v\in\mathrm{Harm}(\Omega)$, and, by symmetry,
$\tilde f\in A(\Omega\setminus\mathbb{R})$. Moreover, $\tilde u(z)=\begin{cases} u(z), & z\in\Omega^+\cup\sigma\\ u(\bar z), & z\in\Omega^-\end{cases}$

Let us consider $x_0\in\sigma$ and $B(x_0,r)$ as above.

Then $\tilde v\in\mathrm{Harm}(B(x_0,r))$, so $\exists\, U\in\mathrm{Harm}(B(x_0,r))$: $F=U+i\tilde v\in A(B(x_0,r))$.

since $F-f=U-u\in\mathbb{R}$ in $B(x_0,r)\cap\mathbb{H}$, $U-u\equiv\mathrm{const}$. By subtracting it

we get $U=u$ in $B(x_0,r)\cap\mathbb{H}$.

Consider $W=U(z)-U(\bar z)$. On σ, $W(x)=U(x)-U(x)=0$, so

on σ, $\frac{\partial W}{\partial x}=0$. But for $x\in\sigma$, $\frac{\partial W}{\partial y}=2\frac{\partial U}{\partial y}\underset{\text{Cauchy}}{=}-2\frac{\partial v}{\partial x}=0.$

Consider $W = U(\bar{z}) - U(z)$ on σ, $W(x) = U(x) - U(x) = 0$, so

on σ, $\frac{\partial W}{\partial x} = 0$. But for $x \in \sigma$, $\frac{\partial W}{\partial y} = 2\frac{\partial U}{\partial y} \underset{\text{Cauchy Riemann}}{=} -2\frac{\partial v}{\partial x} = 0$.

So the analytic function $\frac{\partial W}{\partial x} - i\frac{\partial W}{\partial y} = 0$ on $\sigma \Rightarrow$

$W \equiv 0 \Rightarrow U(\bar{z}) = U(z) = u(z)$. So $f = F \in A(\Omega)$ ∎

Remark. We can map the real line to any line or circle by Möbius map. So we can get an invariant form of the reflection principle:

Theorem Let C_1, C_2 be two circles or lines

Let Ω be symmetric with respect to C_1, with components of $\Omega \setminus C_1$ denoted by Ω^+ and Ω^-. Let $\sigma := \partial\Omega^+ \cap C_1$.

If $f \in A(\Omega^+) \cap C(\Omega^+ \cup \sigma)$ and $\forall\, x \in \sigma$, $f(x) \in C_2$.

$$\tilde{f}(z) = \begin{cases} f(z), & z \in \Omega^+ \cup \sigma \\ f(z^*)^{**}, & z \in \Omega^- \end{cases} \in A(\Omega).$$

Here z^* is the point symmetric to z wrt C_1

w^{**} is the point symmetric to w wrt C_2.

Proof. Let S_1, S_2 be Möbius maps such that $S_1(C_1) = \mathbb{R}$, $S_2(C_2) = \mathbb{R}$.

Then $S_1(\Omega)$ is symmetric wrt $\mathbb{R}$.

$g = S_2 \circ f \circ S_1^{-1}$ on $S_1(\Omega^+)$ satisfies the conditions of Schwarz Theorem.

So it can be continued to $S_1(\Omega^-)$ by $g(z) = \overline{g(\bar{z})}$.

Then $\tilde{f} = S_2^{-1} \circ g \circ S_1 \in A(\Omega)$, $\tilde{f}(z) = f(z^*)^{**}$, since Möbius maps preserve symmetry.

Another approach: removability.

Theorem (removability). Let $f \in A(\Omega \setminus \mathbb{R}) \cap C(\Omega)$.

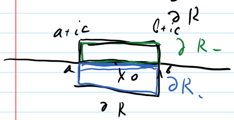
Then $f \in A(\Omega)$.

Removability $\Rightarrow$ Schwarz. The function $F(z)$ satisfies the conditions.

Remark. Removability Theorem also holds for $f \in A(\Omega \setminus C_1) \cap C(\Omega)$ where C_1 - a circle or line.

Proof (of removability). Need only to prove analyticity in $\Omega \cap \mathbb{R}$.

By Morera, it is enough to prove that $\forall B(x_0,r)\subset\Omega$, $x_0\in\mathbb{R}$, and $\forall R\subset B(x_0,r)$ - rectangle with the sides parallel to coordinate axes, $\oint f(z)\,dz=0$.



But let $R_+ = \{z\in R;\ \operatorname{Im} z\geq 0\}$ - upper part
$R_- = \{z\in R;\ \operatorname{Im} z\leq 0\}$ - lower part.

Then $\oint_{\partial R} f(z)\,dz = \oint_{\partial R_+} f(z)\,dz + \oint_{\partial R_-} f(z)\,dz$.

Let $R_+^\varepsilon = \{z\in R, \operatorname{Im} z\geq \varepsilon\}$. Then $R_+^\varepsilon\subset B(x_0,r)\setminus\mathbb{R}\subset\Omega\setminus\mathbb{R}$, so since $f\in A(B(x_0,r)\setminus\mathbb{R})$, $\oint_{\partial R_+^\varepsilon} f(z)\,dz=0$.

Let R_+ has vertices $a, b, b+ic, a+ic$.

Then $\oint_{\partial R_+} f(z)\,dz = \int_a^b f(x)\,dx + \int_0^c f(b+iy)\,dy - \int_a^b f(x+ic)\,dx - \int_0^c f(iy)\,dy$

$\int_{\partial R_+^\varepsilon} f(z)\,dz = \int_a^b f(x+i\varepsilon)\,dx + \int_\varepsilon^c f(b+iy)\,dy - \int_a^b f(x+ic)\,dx - \int_\varepsilon^c f(iy)\,dy$

But $f(x+i\varepsilon)\to f(x)$ uniformly on $[a,b]$ - uniform continuity of f.

$\int_\varepsilon^c \to \int_0^c$ for any continuous function.

So $\oint_{\partial R_+^\varepsilon} \to \oint_{\partial R_\varepsilon} f(z)\,dz$, thus $\oint_{\partial R_+} f(z)\,dz=0$.
Same reasoning, $\oint_{\partial R_-} f(z)\,dz=0$.

$\Rightarrow \oint_{\partial R} f(z)\,dz=0$